DOMAIN WALL SOLUTIONS

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1. Introduction

Of the large variety of topological defects that are known, domain walls are the simplest to understand in many ways. The underlying topology that gives rise to domain walls is discrete and can easily be visualized. In some models the equations of motion can be solved analytically leading to explicit expressions. Since the topology is discrete, formation via the Kibble mechanism can be discussed in terms of known results in percolation theory. Hence it is instructive to embark on a study of topological defects starting with the simplest case of domain walls. The purpose of these lectures is to get you started.

As often happens, the simplest of considerations evolve into more complex phenomena. And domain walls are no exception. Only in the simplest of models are domain walls simple. As we go to somewhat more elaborate and realistic models, the domain walls in them get more complicated and realistic! Some of the developments that I will describe have direct connection with the investigation of domain walls in He-3 [1]. In these lectures I will develop the subject enough so that you can see the complexities emerging. However, there will not be enough time to cover all the recent advances and those woods will be left for you to explore on your own. An important part of the forest that I will not get into is the cosmology of domain walls. This includes discussion of their dynamics and cosmological evolution. There simply isn't enough time for it. I am hoping that the lectures by Professor Arodz will rectify some of these omissions.

I will begin by describing the simplest of domain walls. This is the "kink" in a Z_2 model. Then I will consider the more complicated field theoretic model based on $SU(5)\times Z_2$ symmetry. (The discussion is easily carried over to the case of $SU(N)\times Z_2$ for odd N.) The discrete symmetry responsible for domain walls is still Z_2 like in the simple model. Here, however, we will find a discrete spectrum of domain walls all having the same topology but different masses and other properties. The formation of domain walls in this model via the Kibble mechanism and the importance of these considerations in connection with cosmology will be discussed.

2. The kink

Consider the ϕ^4 Lagrangian in 1+1 dimensions labeled by (t, z)

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4}(\phi^2 - \eta^2)^2 \tag{1}$$

where $\phi(t, z)$ is a real scalar field – also called the order parameter. The Lagrangian is invariant under $\phi \to -\phi$ and hence possesses a Z_2 symmetry. For this reason, the potential has two minima: $\phi = \pm\eta$, and the "vacuum manifold" has two-fold degeneracy.

Consider the possibility that $\phi = +\eta$ at $z = +\infty$ and $\phi = -\eta$ at $z = -\infty$. In this case, the continuous function $\phi(z)$ has to go from $-\eta$ to $+\eta$ as z is taken from $-\infty$ to $+\infty$ and so must necessarily pass through $\phi = 0$. But then there is energy in this field configuration since the potential is non-zero when $\phi = 0$. Also, this configuration cannot relax to either of the two vacuum configurations, say $\phi(z) = +\eta$, since that involves changing the field over an infinite volume from $-\eta$ to $+\eta$, which would cost an infinite amount of energy.

Another way to see this is to notice the presence of a conserved current:

$$\eta j^\mu = \epsilon^{\mu\nu} \partial_\nu \phi$$

where $\mu, \nu = 0, 1$ and $\epsilon^{\mu\nu}$ is the antisymmetric symbol in 2 dimensions. Clearly j^μ is conserved and so we have a conserved charge in the model:

$$\eta Q = \int dz j^0 = \phi(+\infty) - \phi(-\infty) \ .$$

For the vacuum $Q = 0$ and for the configuration described above $Q = 1$. So the configuration cannot relax into the vacuum - it is in a different topological sector.

To get the field configuration with the boundary conditions $\phi(\pm\infty) = \pm\eta$, one would have to solve the field equation resulting from the Lagrangian (1). This would be a second order differential equation. Instead, one can use the clever method first derived by Bogomolnyi [2] and obtain a first order differential equation. The method uses the energy functional:

$$E \ = \ \int dz \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_z \phi)^2 + V(\phi) \right]$$

$$= \int dz \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_z \phi - \sqrt{2V(\phi)}\ )^2 + \sqrt{2V(\phi)}\partial_z \phi \right]$$
$$= \int dz \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_z \phi - \sqrt{2V(\phi)}\ )^2 \right] + \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi' \sqrt{2V(\phi')}$$

Then, for fixed values of ϕ at $\pm\infty$, the energy is minimized if

$$\partial_t \phi = 0$$

and

$$\partial_z \phi - \sqrt{2V(\phi)} = 0\ .$$

Furthermore, the minimum value of the energy is:

$$E_{min} = \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi' \sqrt{2V(\phi')}\ .$$

In our case,

$$\sqrt{V(\phi)} = \sqrt{\frac{\lambda}{4}}(\eta^2 - \phi^2)$$

which can be inserted in the above equations to get the "kink" solution:

$$\phi = \eta\ \tanh\left(\sqrt{\frac{\lambda}{2}}\eta z \right)$$

for which the energy per unit area is:

$$\sigma_{kink} = \frac{2\sqrt{2}}{3}\sqrt{\lambda}\eta^3 = \frac{1}{3}\frac{m^3}{\sqrt{\lambda}} \tag{2}$$

where $m = \sqrt{2\lambda}\eta$ is the mass of excitations (particles) of ϕ in the vacuum of the model. Note that the energy density is localized in the region where ϕ is not in the vacuum, *i.e.* in a region of thickness $\sim m^{-1}$ around $z = 0$.

We can extend the model in eq. (1) to 3+1 dimensions and consider the case when ϕ only depends on z but not on x and y. We can still obtain the kink solution for every value of x and y and so the kink solution will describe a "domain wall" in the $xy-$plane.

At the center of the kink, $\phi = 0$, and hence the Z_2 symmetry is restored in the core of the kink. In this sense, the kink is a "relic" of the symmetric phase of the system. If kinks were present in the universe today, their interiors would give us a glimpse of what the universe was like prior to the phase transition.

3. $SU(5) \times Z_2$ walls

An example that is more relevant to cosmology is motivated by Grand Unification. Here we will consider the $SU(5)$ model:

$$L = \mathrm{Tr}(D_\mu \Phi)^2 - \frac{1}{2}\mathrm{Tr}(X_{\mu\nu}X^{\mu\nu}) - V(\Phi) \tag{3}$$

where, in terms of components, $\Phi = \Phi^a T^a$ is an $SU(5)$ adjoint, the gauge field strengths are $X_{\mu\nu} = X^a_{\mu\nu}T^a$ and the $SU(5)$ generators T^a are normalized such that $\mathrm{Tr}(T^a T^b) = \delta^{ab}/2$. The definition of the covariant derivative is:

$$D_\mu \Phi = \partial_\mu \Phi - ie[X_\mu, \Phi] \tag{4}$$

and the potential is the most general quartic in Φ:

$$V(\Phi) = -m^2\mathrm{Tr}(\Phi^2) + h[\mathrm{Tr}(\Phi^2)]^2 + \lambda\mathrm{Tr}(\Phi^4) + \gamma\mathrm{Tr}(\Phi^3) - V_0 \ , \tag{5}$$

where, V_0 is a constant that we will choose so as to set the minimum value of the potential to zero.

The $SU(5)$ symmetry is broken to

$$H = [SU(3) \times SU(2) \times U(1)]/Z_6 \tag{6}$$

if the Higgs acquires a VEV equal to

$$\Phi_0 = \frac{\eta}{2\sqrt{15}}\mathrm{diag}(2, 2, 2, -3, -3) \tag{7}$$

where

$$\eta = \frac{m}{\sqrt{\lambda'}} \ , \quad \lambda' \equiv h + \frac{7}{30}\lambda \ . \tag{8}$$

For the potential to have its global minimum at $\Phi = \Phi_0$, the parameters are constrained to satisfy:

$$\lambda \geq 0 \ , \quad \lambda' \geq 0 \ . \tag{9}$$

For the global minimum to have $V(\Phi_0) = 0$, in eq. (5) we set

$$V_0 = -\frac{\lambda'}{4}\eta^4 \ . \tag{10}$$

The model in eq. (3) with the potential in eq. (5) does not have any topological domain walls because there are no broken discrete symmetries. In particular, the Z_2 symmetry under $\Phi \to -\Phi$ is absent due to the cubic term. However if γ is small, there are walls connecting the two vacua related by $\Phi \to -\Phi$ that are almost topological. In our analysis we will set $\gamma = 0$, in which case the symmetry of the model is $SU(5) \times Z_2$. An expectation of Φ breaks the Z_2 symmetry and leads to topological domain walls.

The Lagrangian in eq. (3) provides us with the (second order) equations of motion for all the fields. However, it does not tell us the boundary conditions on the fields. The boundary conditions will depend on the class of solutions that we are interested in. This is the first issue that we need to settle.

We would like a solution that has the correct topology. Topology tells us that the vacuum manifold is disconnected. One of the disconnected regions is described by $-U^\dagger \Phi_0 U$ and the other by $+U^\dagger \Phi_0 U$, where $U \in SU(5)$. To obtain solutions we need $\Phi_- \equiv \Phi(-\infty)$ to be in the first sector and $\Phi_+ \equiv \Phi(+\infty)$ to be in the second disconnected sector. Since we can globally rotate the fields by any $SU(5)$ transformation, we are free to set $\Phi_- = -\Phi_0$. But we still have to choose $\Phi_+ = +U^\dagger \Phi_0 U$ for some $U \in SU(5)$.

The freedom to choose Φ_+ is greatly reduced by the following result: a static solution to the field equations is only possible if $[\Phi_-, \Phi_+] = 0$. I will not give the proof here but you can find it in the Appendix of Ref. [4].

The condition $[\Phi_-, \Phi_+] = 0$ still permits a large variety of boundary conditions Φ_+. But the different choices for Φ_+ can all be rotated into a diagonal choice by a transformation in the unbroken global symmetry group which leaves Φ_- invariant. Hence we need only consider a discrete set of choices for Φ_+, and these are:

$$\Phi_+^{(0)} = \epsilon \frac{\eta}{2\sqrt{15}} \mathrm{diag}(2, 2, 2, -3, -3) \tag{11}$$

$$\Phi_+^{(1)} = \epsilon \frac{\eta}{2\sqrt{15}} \mathrm{diag}(2, 2, -3, 2, -3) \tag{12}$$

$$\Phi_+^{(2)} = \epsilon \frac{\eta}{2\sqrt{15}} \mathrm{diag}(2, -3, -3, 2, 2) \tag{13}$$

The superscript on Φ_+ corresponds to the number of entries of Φ_- that have been permuted. The parameter ϵ has the value -1 for the topological domain walls. Later we will find that non-trivial solutions also exist for $\epsilon = +1$. These correspond to non-topological domain walls.

It is tempting to try to obtain the domain wall solutions using Bogomolnyi's method. This has not been done and remains an interesting open problem. However, we can write down the solution for the zero permutation case since it is exactly like the Z_2 case:

$$\Phi^{(0)} = \tanh(X) \Phi_0 \tag{14}$$

where $X \equiv mx/\sqrt{2}$.

The other cases are a little more involved but still quite simple. The solution is of the form:

$$\Phi_k = F_+(x)\mathbf{M}_+ + F_-(x)\mathbf{M}_- + g(x)\mathbf{M} \ , \tag{15}$$

where

$$\mathbf{M}_+ = \frac{\Phi_+ + \Phi_-}{2} \ , \quad \mathbf{M}_- = \frac{\Phi_+ - \Phi_-}{2} \ , \tag{16}$$

$g(\pm\infty) = 0$ and $\mathbf{M}$ is yet to be found.

It is now convenient to move from the $SU(5)$ to the $SU(N)$ case. Let $N = 2n+1$, so $n = 2$ when $N = 5$. Also let us label the number of permutations in the boundary conditions by the integer q. For $N = 5$, the three possible boundary conditions in eq. (11)-(13) correspond to $q = 0, 1, 2$. The advantage of dealing with the general case is that we can write one equation in terms of n and q rather than 3 equations, one per case.

The formulae for $\mathbf{M}_\pm$ can now be explicitly written (we set $\epsilon = -1$ for now):

$$\mathbf{M}_+ = \eta N \sqrt{\frac{1}{2N(N^2-1)}} \mathrm{diag}(\mathbf{0}_{n+1-q}, \mathbf{1}_q, -\mathbf{1}_q, \mathbf{0}_{n-q}) \ , \tag{17}$$

$$\mathbf{M}_- = \eta \sqrt{\frac{1}{2N(N^2-1)}} \mathrm{diag}(-2n\mathbf{1}_{n+1-q}, \mathbf{1}_q, \mathbf{1}_q, 2(n+1)\mathbf{1}_{n-q}) \ . \tag{18}$$

We have used $\mathbf{0}_k$ and $\mathbf{1}_k$ to denote the $k-$dimensional zero and unit matrices respectively. Note that the matrices $\mathbf{M}_\pm$ are orthogonal:

$$\mathrm{Tr}(\mathbf{M}_+\mathbf{M}_-) = 0 \ , \tag{19}$$

but are not normalized to $1/2$. The boundary conditions for $F_\pm$ are:

$$\begin{aligned} F_-(-\infty) &= -1 \ , \quad F_-(+\infty) = +1 \ , \\ F_+(-\infty) &= +1 \ , \quad F_+(+\infty) = +1 \ . \end{aligned} \tag{20}$$

The advantage of this form of the ansatz is that, for particular values of the parameters of a quartic potential in the $q = n$ topological ($\epsilon = -1$) case, one finds the explicit and simple solution $F_+(x) = 1$, $F_-(x) = \tanh(\sigma x)$ and $g(x) = 0$, where $\sigma = mx/\sqrt{2}$ [5, 4]. Also, for $q = 0$, $\epsilon = -1$, the solution is the embedded Z_2 kink *i.e.* $F_+(x) = g(x) = 0$, $F_-(x) = \tanh(\sigma x)$.

Now we would like to find the unknown matrix $\mathbf{M}$ in the ansatz (15). By inserting the ansatz in the equations of motion [6], it turns out that $\mathbf{M}$ is uniquely determined and this is:

$$\begin{aligned} \mathbf{M} = \mu \, \mathrm{diag} \quad (& \quad q(n-q)\mathbf{1}_{n+1-q}, -(n-q)(n+1-q)\mathbf{1}_q, \\ - & \quad (n-q)(n+1-q)\mathbf{1}_q, q(n+1-q)\mathbf{1}_{n-q}) \end{aligned} \tag{21}$$

with μ being a normalization factor in which we also include the energy scale η for convenience:

$$\mu = \eta[2q(n-q)(n+1-q)\{2n(n+1-q)-q\}]^{-1/2} \ . \tag{22}$$

Note that the matrix $\mathbf{M}$ is not normalizable if $q = 0$ or if $q = n$. For these values of q, we can set $g(x) = 0$.

The functions $F_{\pm}(x)$ and $g(x)$ can be found by solving their equations of motion derived from the Lagrangian together with the specified boundary conditions. There is no guarantee that a solution will exist and so we find the solutions explicitly for $N = 5$ with a quartic potential. The solution for $q = 0$ is simply that in eq. (14). The $q = 1$ profile functions are evaluated numerically and shown in Figure 1. The profiles for $q = 2$ have also been evaluated numerically in Ref. [5]. However, an interesting situation arises for $q = 2$ when

$$\frac{h}{\lambda} = -\frac{3}{N(N-1)} = -\frac{3}{20} \tag{23}$$

For this special value of coupling constants, the profile functions can be written as [4]:

$$F_+ = 1 \ , \quad F_- = \tanh(X) \ , \quad g = 0 \tag{24}$$

It is not understood what is special about $h/\lambda = -3/20$ that it allows an analytic solution. Could it be that there is a symmetry of the system at this point in parameter space that simplifies matters? Could a Bogomolnyi type analysis be done at this point?

The stability of these solutions has been investigated in Ref. [5, 4, 6] and the $q = 2$ solution turns out to be the minimum energy domain wall solution. For the particular choice of model parameters given in eq. (23) (valid for $N > 3$) [4], the energy of the $q = 2$ kink is:

$$\sigma = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}\left(\frac{N-1}{N-3}\right) \ . \tag{25}$$

An interesting point to note is that the ansatz is valid even if $\Phi_{\pm}$ are not in distinct topological sectors *i.e.* even if $\epsilon = +1$. These imply the existence of non-topological kink solutions in the model. If we include a subscript NT to denote "non-topological" and T to denote "topological", we have

$$\Phi_{NTk} = F_+(x)\mathbf{M}_{NT+} + F_-(x)\mathbf{M}_{NT-} + g(x)\mathbf{M}_{NT} \ . \tag{26}$$

Since $\Phi_{NT+} = -\Phi_{T+}$, we find

$$\mathbf{M}_{NT+} = \mathbf{M}_{T-} \ , \quad \mathbf{M}_{NT-} = \mathbf{M}_{T+} \ , \quad \mathbf{M}_{NT} = \mathbf{M}_{T} \ . \tag{27}$$

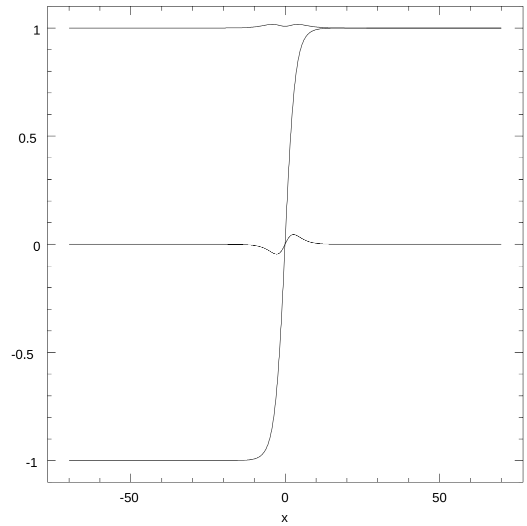

Figure 1. The profile functions F_+ (nearly 1 throughout), F_- (shaped like a tanh function), and g (nearly zero) for the $q = 1$ topological kink with parameters $h = -3/70$, $\lambda = 1$ and $\eta = 1$.

Hence

$$\Phi_{NTk} = F_-(x)\mathbf{M}_{T+} + F_+(x)\mathbf{M}_{T-} + g(x)\mathbf{M}_T \ . \tag{28}$$

So to get F_- (F_+) for the non-topological kink we have to solve the topological F_+ (F_-) equation of motion with the boundary conditions for F_- (F_+). To obtain g for the non-topological kink, we need to interchange F_+ and F_- in the topological equation of motion. The boundary conditions for g are unchanged. Generally the non-topological solutions, if they exist, will be unstable. However, the possibility that some of them may be locally stable for certain potentials cannot be excluded.

3.1. DOMAIN WALL LATTICE

The forces between domain walls of different kinds has been studied by Pogosian [7]. What is most interesting is that the $q = 2$ wall and anti-wall can repel. Here one needs to be careful about the meaning of an "anti-wall". An anti-wall should have a topological charge that is opposite to that of a wall. That is, a wall and its anti-wall together should be in the trivial topological sector. But this condition still leaves open several different kinds of anti-walls for a given wall. To be specific consider a domain wall across which the Higgs field changes from $(2, 2, 2, -3, -3)$ to $-(2, -3, -3, 2, 2)$. (I am surpressing the normalization factor for convenience of writing.) There can be two corresponding anti-walls. In the first type – called Type I – the Higgs field can go from $-(2,-3,-3,2,2)$ to $+(2,2,2,-3,-3)$, thus reverting the change across the first domain wall. In the second type (Type II), the Higgs field can go from $-(2,-3,-3,2,2)$ to $+(-3,2,2,-3,2)$. Pogosian found that the force between a wall and its Type I anti-wall is attractive, but the force between a wall and its Type II anti-wall is repulsive. This is understood by noting that the force between walls is proportional to $\mathrm{Tr}(Q_1 Q_2)$ where Q_1 and Q_2 are the charge matrices of the two walls. If the Higgs field to the left side of the walls is Φ_L, between the two walls is Φ_0, and is Φ_R on the right-hand side of the walls, then $Q_1 \propto \Phi_0 - \Phi_L$ and $Q_2 \propto \Phi_R - \Phi_0$. Then the stable ($q = 2$) walls can have charge matrices proportional to $(-4,1,1,1,1)$, $(1,-4,1,1,1)$, $(1,1,-4,1,1)$, $(1,1,1,-4,1)$ and $(1,1,1,1,-4)$. (Stable anti-walls have the same charges but with a minus sign.) Then it is easy to see that one can take a wall with one of the 5 charges listed above and it will repel an anti-wall which has the -4 occurring in some other entry. Hence there are combinations of walls and anti-walls for which the interaction is repulsive.

We know that topology forces a wall to be followed by an anti-wall. Then we can set up a sequence of walls and anti-walls in the following way:

$$...Q^{(1)}\bar{Q}^{(5)}Q^{(3)}\bar{Q}^{(1)}Q^{(5)}\bar{Q}^{(3)}... \tag{29}$$

where Q_i and $\bar{Q}_i$ refer to a wall and an anti-wall of type i respectively. Alternately, this sequence of walls would be achieved with the following sequence of Higgs field expectation values:

$$\begin{aligned} ... \to -(2,2,2,-3,-3) \;&\to\; +(2,-3,-3,2,2) \\ &\to\; -(-3,2,2,-3,2) \\ &\to\; +(2,-3,2,2,-3) \\ &\to\; -(2,2,-3,-3,2) \\ &\to\; +(-3,-3,2,2,2) \\ &\to\; -(2,2,2,-3,-3) \to ... \end{aligned} \tag{30}$$

The forces between walls fall off exponentially fast and hence the dominant forces will be between nearest neighbors.

Note that the sequence of walls is periodic with a period of 6 walls, and these 6 walls have a net topological charge that vanishes. Hence we can put the sequence in a periodic box i.e. compact space. This gives us a finite lattice of domain walls.

The sequence described above is one whose period is the minimum possible (namely, 6). It is easy to construct other sequences with greater periodicity. For example:

$$...Q^{(1)}\bar{Q}^{(5)}Q^{(3)}\bar{Q}^{(4)}Q^{(2)}\bar{Q}^{(1)}Q^{(5)}\bar{Q}^{(3)}Q^{(4)}\bar{Q}^{(2)}... \tag{31}$$

is a repeating sequence of 10 domain walls.

In ongoing work, we have checked that the lattice is a solution of the equations of motion. However, it is not stable. The instability is towards rotations of the walls in internal space. For example, the $\bar{Q}^{(5)}$ antiwall could rotate to become either a $\bar{Q}^{(1)}$ or a $\bar{Q}^{(3)}$ antiwall. Once this rotation takes place, this antiwall will be attracted by the $Q^{(1)}$ or the $Q^{(3)}$ neighbor, and annihilation can occur. An analogy of this process is due to Pogosian – if we place several bar magnets along a line such that the North poles of neighboring magnets face each other, then each bar magnet will be unstable to rotation. Once the bar magnets rotate and North poles face neighboring South poles, the magnets will attract and the chain of magnets is unstable to collapse. In this analogy, the rotation is in physical space; in the case of domain walls, the rotation is in internal space.

The understanding of the domain wall lattice instability, provides us with another avenue of investigation. If the bar magnets were confined to one dimensions (for example, they could be placed in a tube, or attached to a wire), then the instability would cease to exist since rotations would not be possible. The corresponding situation in the case of domain walls is if the unbroken symmetry is too restrictive to allow rotation. This can be achieved by breaking the $SU(5) \times Z_2$ to $U(1)^4$ or something smaller. A simpler alternative is to start out with a model in which there are no rotational degrees of freedom present. This scheme is discussed in [8]. One starts with a model that only has a discrete symmetry group that corresponds to permutations of the diagonal entries of $SU(5)$ and the sign flip given by Z_2. It can be shown that the domain wall lattice is stable in this model.

3.2. FORMATION OF DOMAIN WALLS

The properties of the network of domain walls at formation has been determined by numerical simulations. The idea behind the simulations is that the vacuum in any correlated region of space is determined at random. Then, if there are only two degenerate vacuua (call them + and −), there will be spatial regions that will be in the + phase and others in the − phase. The boundaries between these regions of different phases is the location of the domain walls. This is nothing other than the “Kibble mechanism”.

By performing numerical simulations, the statistics shown in Table I was obtained [9]. The data shows that there is essentially one giant connected + cluster. By symmetry there will be one connected − cluster. In the infinite volume limit, these clusters will also be infinite and their surface areas will

Cluster size	1	2	3	4	6	10	31082
Number	462	84	14	13	1	1	1

also be infinite. Therefore the topological domain wall formed at the phase transition will be infinite.

What does the Kibble mechanism predict for $SU(5) \times Z_2$ domain walls? Once again we have to throw down values of the Higgs field on a lattice and then examine the walls that would form at the interface. We have found that there are 3 kinds of wall solutions and so each one will be formed with some probability. In the Kibble mechanism approach, the probability that a certain wall will form is directly related to the number of boundary values that result in the formation of a defect. So we need to evaluate all the boundary conditions that will lead to domain walls with a certain value of q. In other words, we want to determine the "space of kinks" for a fixed value of q [4, 6].

Let the Higgs field to the left and right of a domain wall be Φ_L and Φ_R respectively. Without loss of generalization, we can take $\Phi_L = \Phi_0$. To start, let us find all values of Φ_R that will give a $q = 0$ kink. But there is only one such value, namely $\Phi_R = -\Phi_0$. So the space of the $q = 0$ kink is just one point. Next, let us determine all Φ_R that will give a $q = 1$ kink. Here there are several possibilities since different entries of Φ_0 can be permuted to give different Φ_R. In fact, one can take any choice of Φ_R, for example along $-(2, 2, -3, 2, -3)$ and act by gauge transformations belonging to the unbroken symmetry group H (see eq. (6)), and this will lead to another choice for Φ_R. Of course, some of these rotations will leave Φ_R unchanged and these should not be counted. So the space of $q = 1$ kinks is given by the coset space:

$$\Sigma_1 = H/K_1 \tag{32}$$

where

$$K_1 = [SU(2) \times U(1)^3]/Z_2 \ . \tag{33}$$

A similar argument shows that the space of $q = 2$ kinks is given by:

$$\Sigma_2 = H/K_2 \ , \tag{34}$$

where

$$K_2 = [SU(2)^2 \times U(1)^2]/Z_2^2 \ . \tag{35}$$

Now that we have the space of kinks for each q, we note that the space of the $q = 0$ kink is zero dimensional, the space of the $q = 1$ kink is 6 dimensional, and the space of the $q = 2$ kink is 4 dimensional. Hence, in the Kibble mechanism approach, the probability of a kink being of the $q = 0$ or $q = 2$ variety is zero, and the probability of the kink being of the $q = 1$ variety is 1.

A subtlety that has not been discussed above is that there is also the possibility that if we lay down Higgs fields randomly, we may get $[\Phi_L, \Phi_R] \neq 0$. In this case, as described earlier, there will be no static solution to the equations. Then the field configuration will evolve towards a static configuration. Our discussion above assumes that such a configuration has been reached, and neighboring domains always have values of Φ that commute. This is not completely satisfactory since there will be time scales that are associated with the relaxation and these must be compared to other time scales characterizing the phase transition.

To summarize, the Kibble mechanism predicts that only $q = 1$ domain walls will be formed at the $SU(5) \times Z_2$ phase transition. However, we know that the stable variety of walls have $q = 2$, and the $q = 1$ walls will decay into them. Precisely how the $q = 2$ walls convert into $q = 1$ walls during a phase transformation has not been studied. Nor is it known how long this relaxation will take. The answers to these questions is of some importance as we will briefly see in the next section.

3.3. IMPORTANCE IN COSMOLOGY

The $SU(5) \times Z_2$ symmetry breaking also leads to topological magnetic monopoles which are cosmologically unacceptable. However these magnetic monopoles interact with the domain walls that we have discussed in the earlier sections. If the domain wall is of the $q = 0$ variety, the monopole is certainly destroyed and its magnetic charge spreads out on the domain wall [3, 5]. If the domain wall is of the $q = 1$ variety, monopoles in a large number of orientations in the group space get destroyed whereas others survive. If the domain wall is of the $q = 2$ variety, a smaller number of monopoles get destroyed [6]. So the cosmological predictions depend on the kinds of walls that are formed and their interactions with magnetic monopoles. This is why it is important to consider the types of domain walls that will be formed and the eventual relaxation of the system, especially in regard to the number density of magnetic monopoles.

References

1. M.M. Salomaa and G.E. Volovik, "Cosmiclike domain walls in superfluid ^{3}He-B: Instantons and diabolical points in $(\vec{k}, \vec{r})$ space", Phys. Rev. B **37**, 9298 - 9311 (1988).
2. E. B. Bogomolnyi, *Sov. J. Nucl. Phys.* **24** 449 (1976); reprinted in "Solitons and Particles", eds. C. Rebbi and G. Soliani (World Scientific, Singapore, 1984).
3. G. Dvali, H. Liu and T. Vachaspati, *Phys. Rev. Lett.* **80**, 2281 (1998).
4. T. Vachaspati, Phys. Rev. **D63**, 105010 (2001).

5. L. Pogosian and T. Vachaspati, Phys. Rev. **D62**, 123506 (2000).
6. L. Pogosian and T. Vachaspati, Phys. Rev. **D64**, 105023 (2001).
7. L. Pogosian, Phys. Rev. **D65**, 065023 (2002).
8. L. Pogosian and T. Vachaspati, hep-th/0210232 (2002).
9. T. Vachaspati and A. Vilenkin, Phys. Rev. **D30**, 2036 (1984).